A REFERENCE PAPER ON PLANAR COLOURED GRAPHS WITH EXAMPLES

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ABSTRACT

After a brief historical account, a few simple structural theorems about plane graphs useful for coloring are stated, and two simple applications of discharging are given. Afterwards, the following types of proper colorings of plane graphs are discussed, both in their classical and choosability (list coloring) versions: simultaneous colorings of vertices, edges, and faces (in all possible combinations, including total coloring), edge-coloriing, cyclic coloring (all vertices in any small face have different colors), 3-coloring, acyclic coloring (no 2-colored cycles), oriented coloring (homomorphism of directed graphs to small tournaments), a special case of circular coloring (the colors are points of a small cycle, and the colors of any two adjacent vertices must be nearly opposite on this cycle), 2-distance coloring (no 2-colored paths on three vertices), and star coloring (no 2-colored paths on four vertices). The only improper coloring discussed is injective coloring (any two vertices having a common neighbor should have distinct colors).

INTRODUCTION AND PRELIMINARIES

Coloring in a broad sense is a decomposition of a discrete object into simpler sub-objects. Due to its generality, this notion arises in various branches of discrete mathematics and has important applications. For example, one of the most natural models in the frequency assignment problem in mobile phoning is L(p,q)- labeling. The vertices of a planar graph (sources) should be colored (get frequencies assigned) so that the colors (integer frequencies) of vertices at distance 1 differ by at least p, while those at distance 2 differ by at least q. Sometimes, the set of available frequencies can vary from one source to another; this corresponds to "list L(p,q)-labeling".

The theory of plane graph coloring has a long history, extendingback to the middle of the 19th century, inspired by the famousFour Color Problem (4CP), which asked if every plane map is 4-colorable. Now it is a broad area of research, with hundreds ofcontributors and thousands of contributions, and so is covered inthissurveyonlypartially.

REDUCIBLECONFIGURATIONS

The basic elements of a plane map are its vertices, edges, and faces. An edge is a closed Jordan curve; its end-points are vertices. A loop joins a vertex to itself; two vertices may be joined by several multiple edges. No edge can have an internal point in common with itself or with another edge. Faces are connected components of the complement of the map. A plane graph is a plane map with neither loops nor multiple edges. All maps considered in this survey are finite.

A set **S** of (usually small) plane graphs, called configurations, is unavoidable for a class **M** of plane maps if every map M in **M** hasa configuration from **S** as a subgraph. In Section 2, we see a few examples of unavoidable sets of configurations (USC in the sequel) for various classes of plane maps. Among USCs, there are both trivial (vertices of degree at most 5) and very complicated (sets of up to fifteen hundred configurations, most of which consist of more than a dozen vertices; such a set was used by Appel and Haken [12]). A statement that **S** is a USC may comprise a nice theorem on the structure of plane graphs, having its own value (if it is formulated in basic terms and has precise numerical parameters). More often, it is a tool for solving a particular coloring problem. Rather rarely a USC combines both these virtues.

A link between USCs and coloring problems has been known for a long time, beginning with Heawood's Five Color Theorem, which is based on the trivial fact that every planar graph has a vertex of degree at most 5. Almost all previously known theorems on planar graph colorings are based on certain USCs (with very few exceptions, like Thomassen's celebrated Five Choosability Theorem [177]). The idea of the method of reducible configurations (MRC) is as follows. Given a type of coloring, a reducible configuration for that type of coloring is a graph that cannot lie in a minimal plane graph that cannot be colored as required. Accordingly, solving a plane graph coloring problem by the MRC is equivalent to finding an unavoidable set of reducible configurations (USRC) for it. To prove that a configuration C is reducible, a typical approach is to attempt to show that every coloring of the boundary of C can be extended to the whole C.

1. A Quadratic Four-Coloring Algorithm.

Define V(G), E(G), F(G) to be the sets of vertices, edges, and faces of a plane graph G.

A normal plane triangulation G is said to be nearly 7-connected if

(1) G has no k-ring for k E {2, 3,4}, where a k-ring is a cycle on k edges with at least 1 vertex in each of its interior and exterior;

(2) G has no non-trivial 5-ring, which is a cycle on 5 vertices with at least 2 vertices in each of its interior and exterior;

(3) G has no bending 6-ring, which is a cycle C on 6 vertices with at least 4 vertices in each of its interior and exterior, and there is a vertex not on C which is adjacent to three consecutive vertices of C.

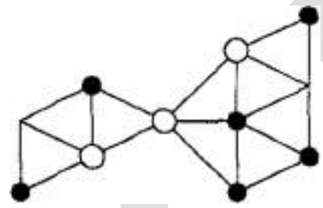
In more detail, let a near-triangulation be a connected plane graph where every finite face has degree three. Then K is a configuration if K is a near triangulation G(K) together with a function -yK from V(G(K)) to the non-negative integers such that properties 1, 2, and 3 below hold. The property of a configuration being reducible will be discussed later. Let the ring-size of K be ZV (YK (v) – deg(v) – 1), summed over all vertices v incident with the infinite face such that G(K) – v is connected.

(1) for every vertex v, G(K) - v has at most two components, and if there are two, then ~~(v) = deg(v) +2;

(2) for every vertex v, if v is not incident with the infinite region, then “/K (w) = deg(v), and otherwise TK (v) > deg(v); and in either case, W(v) 2 5;

(3) the ring-size of K is at least 2.

In a figure, for a vertex v, the value of ~K (v) will be indicated by a shape in the figure. The standard shapes are a solid dot if ~K(v)
= 5, a circle if -yK(v) = 7, a square if ~K (v) = 8, a triangle if ~K
(v) = 9, a pentagon if yK (v) = 10, and no shape if ‘)’K (v) = 6. An example of a configuration of ring size 14 appears in Figure 1,



A configuration.

Let a configuration K be weakly contained in a plane graph G if there is a function ~ from V(G(K)) to V(G) such that

(1) for every v c V(G(K)), deg(~(v)) = ~K(V)

(2) for every v E V(G(K)) such that G(K) - v is connected, if the cyclic neighborhood of v 2S(U1, .!., Uj) such that for every 1 < I

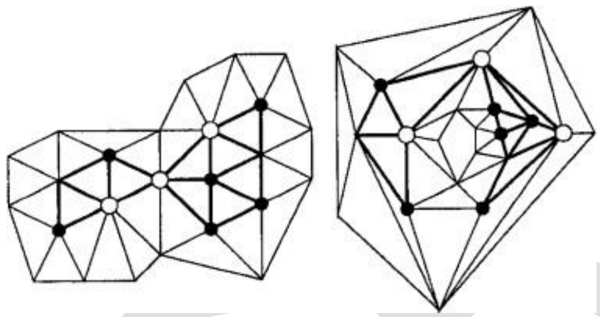
(3) for every v c V(G(K)) such that G(K) – v is not connected, if the cyclic neighborhood of v is (ul, ...uj. vi,..., vk) such that for evey I ~ i < j, Uiui+lv is a triangle of G(K) and for every 1< i< k, v~v,+lv is a triangle of G(K), then the cyclic neighborhood off (v) is (~(”l),..., ~(uj), ~, f(vl), j(vk),x)), x)

Also, let a configuration K be strongly contained in a plane graph G if K is weakly contained in G by means of a function fwhich also satisfies that

(1) for every V,W c V(G(K)), if f(v) = f(zo), then v = w.

(2) for every V, w E V(G(K)), if f(v) is dacent to f(w) in G, thenv is adjacent to w in G(K).

Figure 2 shows the configuration of Figure 1 strongly contained in a plane graph, and also weakly, but not strongly contained ina plane graph.



Strong and weak cent ainment.

THEOREM 1

Every plane triangulation of minimum degree jive weakly contains a configuration in U.

For each vertex x of G, let charge(z) = 10(6 – deg(x)). Since G is a triangulation, a simple manipulation of Euler's formula gives ~Z=v(~) charge(s) = 120. The particular value 120 is not important, just that it is positive. Without modifying the charges, this simply says that every plane triangulation has a vertex of degree at most five. The discharging method works by sending the positive charge away from these vertices, and then, as can be seen by recounting the modified charges, there must still remain positive charge somewhere. Examining the rules that are used to move the charge shows that in each possibility of a vertex x having positive modified charge, the graph has one of the configurations of U present within the second neighborhood of x (meaning the vertices distance at most two from z). Three of the 32 rules that the authors used are as follows:

(1) For each edge st such that deg(s) = 5, send a charge of 2 from s to t.

(2) For each triangle stu such that deg(s) <6, deg(t) >7, and deg(u) = 5, send a charge of 1 from s to t.

(3) For each pair of triangles stu, suv such that deg(s) < 6, deg(t) z 6, deg(u) < 6, and deg(v) = 5, send a charge of 1 from s to t.

Each of the 32 rules sends a charge of 1 (or 2 in just Rule 1) from a source s to a sink t along the edge st dependent only upon the degrees of certain vertices distance at most two from each of s and t. In the 1960s, when it was as yet uncertain that the method of reducible configurations would be able to solve the Four Color Problem, Heesch [7] conjectured that the discharging method could be used to solve it, and further, that the modified charge of a vertex x could be determined by only examining vertices distance at most two from z. The authors have verified this conjecture. This property of the rules is the one that forces any weakly contained configuration K found via discharging to be entirely within the second neighborhood of the vertex whose charge is positive. This in turn shows that if K is not strongly contained in the graph, then either a k-ring for k s 4 or a non-trivial 5-ring is present in the graph.

This is more easily seen using a property of U that was accidental. It turns out that every configuration K of U has diameter at most four; i.e. that each pair of vertices of G(K) have distance at most four in G(K). Using this bound on the diameter, it is easily seen thatif K is weakly, but not strongly contained in G, then G has a k-ring for k≤ 4 or a non-trivial 5-ring. Now the reducibility of configurations will be discussed. Instead of giving a formal definition of reducibility (which can be found in [8]), the properties of reducibility that are needed for the algorithm will be stated.

Given a coloring Ψof a graph G, a pair P of colors, and a vertex x of G colored a color in P, let H(P) be the graph induced by the vertices colored a color in P, and let H(P, x) be the component of H(P) containing z.

Let the coloringΨ′of G obtained from Ψby recoloring each vertex w in II(P, z) with the color in P\{Ψ(u)} becalled the coloring which is Kemped from Ψ

at x by P. This process of recoloring to get from Ψto Ψ′is called a Kempe.

Let K be a configuration of ring-sizer strongly contained in a triangulation G. Let H be the graph obtained from G by deleting the image of G(K); H has a face with a facial walk of length r where the copy of G(K) used to be; call R the vertices incident with thisface. Given a coloring Ψof H with colors in {1, 2,3, 4}, then I!2 can be Kemped into a coloring Φ @ of H if there are an integer j ≥0 and colorings $\Phi_0,\Phi_1,...,\Phi_j$ of H such that $\Phi_0=\Psi,\Phi_j=\Phi$ and for i ∈ {1,. ... j},Φ_i is Kemped from Φ_{i-1} at some vertex of R by some pair of colors in {1,2,3,4}.

Finally, a configuration K of ring-size r is reducible if for every triangulation G strongly containing K, there is a set T of at most four edges of G, each incident with an image of a vertex of G(K), such that if J is obtained from G by contracting the edges of T, then J is loopless, and for every coloring Φ of J,the coloring of G \ G(K) induced by Ψcan be Kemped into a coloring Φ which is extendable into a coloring of G. Moreover, if Ψ is given, Φcan be found by performing at most 3^{2r} Kempes.

Remark: The restriction on the size of T is not important for the algorithm. Also, it was easier for the presentation here to let T depend on G. This does not appear in the true definition of reducibility; the edges are chosen from looking at K only, not G. In fact, the true definition includes no reference to a graph G at all. Checking whether a configuration is reducible only requires the manipulation of colorings of R (whose size is at most r, regardless of G) that do not extend into G(K). This is a finite problem.

It has been claimed that the 633 configurations of U previously mentioned were all reducible. Many configurations of small ring-size have been shown to be reducible by hand [see, e.g. 3], but most configurations of large ring-size have been shown to be reducible by means of a computer program. The C program that the authors used, as well as the configurations in appropriate form for input and a paper explaining thedetails of the program, is available by anonymous ftp from f tp. math. gatech. edu and can be found in the directory pub/users /thomas/f our.

Now the algorithm can be described.

ALGORITHM 1

The input to the algorithm will be a normal plane graph G with n vertices. The output will be a coloring of the vertices of G with four colors.

If $n \leq 4$, just color each vertex a different color. Clearly this uses only constant time.

If G has a face F of degree at least four, then by planarit y, there are two non-adjacent vertices x, y incident with F. Create H from G by identifying x and y into a vertex z, and recurse on H. Color G by coloring each of x, y the color of z, and the other vertices receive the colors they received in H.

If G has a vertex x of degree k ≤4, then the circuit C surrounding it is a k-ring. Go to the k-ring analysis below.

Otherwise G has minimum degree five. By Theorem 1, G must weakly contain a configuration in U. Find a configuration K in U that is weakly contained in G. There are several linear algorithms to perform this, as each vertex of a configuration in U has degree at most 11. If K is not strongly contained in G, then let C be a k-ring for k ≤4 or a non-trivial 5-ring in G. Go to the k-ring analysis below.

Otherwise K is strongly contained in G. Suppose K is a configuration of ring-size r. Every configuration in U has r ≤ 14. Let H be the graph obtained from G by deleting the image of G(K). Let R be the set of vertices, and T be a set of edges as in the definition of reducibility, and then let J be the graph obtained from G by contracting T. Recurse on J. This induces a coloring Ψof H. Kempe the vertices of R until a coloring Φ is found that extends into K. Since K is reducible, Φ can be found by performing at most 3^{2r}Kempes, each of which takes linear time.

Given a ring R which is either a k-ring for k ≤ 4 or a non-trivial 5-ring, a procedure developed by Birkhoff [3] can be used. Let E, 1 be the subgraphs of G on the exterior and interior of R. First form a suitable graph HI from G by deleting 1!?,and performing simple operations to triangulate it (see one of [1, 3, 8] for details). Recurse on HI. This induces a coloring of G \ E. Then form a suitable graph Hz from G by deleting 1, and performing simple operations (which in this case depend on the coloring of HI) to triangulate it, Recurse on H2. This induces a coloring of G \ I. Birkhoff proved that the colorings of G \ E and G \ I can be Kemped to match on R. For the analysis, it is important to know that the simple operations performed to create H_1 and H_2 are such that $|V(H_1)| < n$, $|V(H_2)| < n$, and $|V(H_1)| + |V(H_2)| \leq n+6$

PLANAR GRAPHS ARE 8-ADMISSIBLE

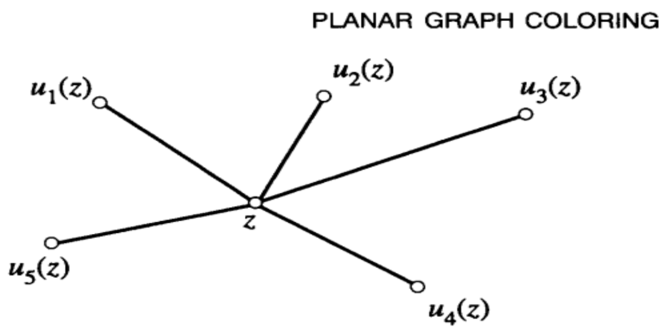
THEOREM 2

Let G = (V, E) be a planar graph. Then there is a linear order L of node set V that has back degree at most 5, admissibility at most 8 and arrangeability at most 10.

Proof

Fix a plane drawing of G that has no edge crossings. We will define the linear order L as a labeling x_1, X_2,... , X_n of the nodes in V. The definition proceeds in reverse order and begins with the choice of X_n as a node of degree at most five in G. Note that the admissibility of X_n is at most 5 and the arrangeability of X_n is O.

At step i, we assume that we have chosen nodes X_{i+1},X_{i+2}, ... ,X_n , and that each of these nodes have back degree at most 5, admissibility at most 8 and arrangeability at most 10. Next, we describe how the node Xi is chosen. The ordering on the nodes x_1,X_2,...,$X_6$is arbitrary so we may assume that i > 7.



Planar graph coloring

CONCLUSION

Two obvious open problems that remain are to tighten the bounds we have produced for the game chromatic number of planar and outerplanar graphs. We have recently shown that the game chromatic number of the class of planar graphs is at least 8, and by the results presented in this paper, it is at most 33. For the class of outerplanar graphs, our bounds are 6 and 8.

More generally, it seems to us to make good sense to investigate general classes of optimization problems that exhibit the key features of the uncooperative (adversarial) graph coloring problem we have studied in this paper.

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